

AN OPERADIC APPROACH TO OPERATOR-VALUED FREE CUMULANTS

GABRIEL C. DRUMMOND-COLE

ABSTRACT. An operadic framework is developed to explain the inversion formula relating moments and cumulants in operator-valued free probability theory.

1. INTRODUCTION

The purpose of this paper is to provide a convenient operadic framework for the cumulants of free probability theory.

In [DCPT15a, DCPT15b], the author and his collaborators described an operadic framework for so-called Boolean and classical cumulants. In those papers, the fundamental object of study is an algebra A equipped with a linear map E , called *expectation*, to some fixed algebra B . The expectation is not assumed to be an algebra homomorphism; rather one measures the degree to which E fails to be an algebra homomorphism with a sequence of multilinear maps κ_n from powers of A to B , called cumulants. The cumulants, in many cases, can be defined recursively in terms of the expectation map via a formula of the form:

$$(1) \quad E(x_1 \cdots x_n) = \sum \kappa_{i_1}(\cdots) \cdots \kappa_{i_k}(\cdots).$$

Depending on what kind of probability theory is under consideration, the summation on the left may be over a different index set. See, e.g., [Spe97, Mur02, HS11].

In [DCPT15a, DCPT15b], these recursive definitions for the collection of cumulants (in the Boolean and classical regimes, respectively) were reinterpreted as the collection of linear maps determining a coalgebraic map into a cofree object. In the Boolean case, the cofree object is the tensor coalgebra. In the classical case it is the symmetric coalgebra.

This reformulation is intended as the background for a homotopical enrichment of probability theory; adding a grading, a filtration, and a differential to this coalgebraic picture leads to a rich theory with applications to quantum field theory [Par15]. This application is motivational and will play no role in this paper.

None of the work mentioned above treats the case of *free cumulants*, arguably the most important kind of cumulant in noncommutative probability theory. When the target algebra B is commutative, there is a formula similar to those above and the framework outlined above can be used directly, employing a more exotic type of coalgebra than the tensor or symmetric coalgebra. This point of view is taken in [DC16].

However, there is a flaw in this point of view, which is that assuming the target to be commutative is external to the theory; internally it makes perfect sense for the

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target itself to be noncommutative. This is called *operator-valued* free probability theory because the expectation is valued in a noncommutative algebra, such as an operator algebra.

Operator-valued free cumulants, as defined by Speicher, [Spe98] are somewhat more cumbersome to describe explicitly than in the commutative case using classical combinatorial methods. Consequently Speicher develops an *operator-valued R -transform* to collect the information concisely.

In our setting, there is one evident related obstruction to extending the framework developed in [DCPT15a, DCPT15b] to operator-valued free cumulants. The defining formulas for classical and Boolean cumulants and for free cumulants valued in a commutative algebra share a certain property. Namely, they are *string-like*, meaning that the right-hand side of Equation (1) is a product of cumulants. However, the defining formulas for operator-valued free cumulants (that is, free cumulants valued in a not necessarily commutative algebra) contain terms like

$$\kappa_2(x_1 \kappa_1(x_2) \otimes x_3).$$

or more generally

$$\kappa_{n_1}(x_1 \kappa_{n_2}(x_2 \kappa_{n_3}(\cdots) \kappa_{n_4}(\cdots), \cdots), \cdots).$$

In a word, they are not string-like but *tree-like*.

This is precisely the issue that leads Speicher to develop the operator-valued R -transform. Here, this tool is avoided by using an operadic reformulation. Tree-like formulas can be obtained by passing from algebras and coalgebras, which have a string-like structure, to nonsymmetric operads and cooperads, which have a tree-like structure. The main result of this paper shows how the relationship between the moments and free cumulants, realized as cooperadic maps M and K , is encapsulated quite simply in terms of a canonical twist:

$$M = \Phi \circ K.$$

As phrased in this paper, the moments and cumulants are defined in some other manner and this is a theorem, but it is probably better to consider this as an alternative definition which is quite simple from the operadic viewpoint.

This reformulation is part of a campaign to explore applications of the operadic language in probability theory; the result contained herein is modest and is intended to serve as further advertisement and evidence (following [Mal11, DCPT15a, DCPT15b, DCT14, DC16]) of potentially deeper connections between the two areas.

It is possible that both this reformulation and those attempted in the author's previous work (cited above) are reflections of a combinatorial relationship between operads and Möbius inversion with respect to a poset. This is not pursued further here, but see [Mén10, 3.3] for some discussion and further references on this topic.

The remainder of the paper is organized as follows. Section 2 reviews the parts of operadic theory that are used in the paper. Section 3 goes over the combinatorics of non-crossing partitions, and Section 4 applies this to define free cumulants. Finally, Section 5 states and proves the reformulation of free cumulants in operadic terms.

1.1. Conventions. Everything linear occurs over a fixed ground ring. Algebras are generally not assumed to be commutative or unital. Every finite ordered set is canonically isomorphic to $[n] := \{1, \dots, n\}$, and this canonical isomorphism will be routinely abused.

A graph is a finite set of vertices, a finite set of half-edges, a source map from half-edges to vertices, and an involution on the half-edges; a half-edge is a leaf if it is a fixed point of the involution. A graph is connected if every two vertices can be joined by a path of half-edges connected by having the same source or via the involution. A connected graph is a tree if it has more vertices than edges. A root is a choice of leaf of a tree (this is no longer considered a leaf). The root of a vertex is the unique half-edge “closest” to the overall root. The root vertex is the unique vertex whose root is the overall root. A planar tree has a cyclic order on the half-edges of each vertex.

2. OPERADS AND COOPERADS

Aside from some minor changes, conventions of [LV12] are used for operadic algebra. This section reviews standard definitions (more details can be seen in [LV12, 5.9]).

Definition 2.1. A *collection* $M = \{M_n\}_{n \geq 0}$ is a set of modules indexed by non-negative numbers (the index is called *arity*).

Given a collection M , a graph *decorated* by M is a pair (G, D) where G is a graph and $D = \{D_v\}$ is a collection of elements of M indexed by the vertices of G ; for a vertex of valence $k + 1$ the decoration D_v should be in the module M_k .

There is a *composition product* denoted \circ on collections

$$(M \circ N)_n = \bigoplus_k M_k \otimes \left(\bigoplus_{i_1 + \dots + i_k = n} N_{i_1} \otimes \dots \otimes N_{i_k} \right).$$

This product has a unit I , where I_1 is the ground ring and $I_{n \neq 1}$ is 0, and together \circ and I make the category of collections into a monoidal category.

Definition 2.2. A *nonsymmetric operad* is a monoid \mathbf{P} in this monoidal category. Its data can be specified by giving a collection P , a *composition* map $\gamma : P \circ P \rightarrow P$, and a *unit* map $\eta : I \rightarrow P$ satisfying associativity and unital constraints.

A *nonsymmetric cooperad* is a comonoid \mathbf{C} in this monoidal category. Its data can be specified by giving a collection C , a *decomposition* map $\Delta : C \rightarrow C \circ C$, and a *counit* map $\epsilon : C \rightarrow I$ satisfying coassociativity and counital constraints. The collection I has a canonical nonsymmetric cooperad structure, denoted \mathbf{I} .

In this paper everything will be nonsymmetric and the adjective will be omitted.

Definition 2.3. Let \mathbf{C} be a cooperad with underlying collection C . A *coaugmentation* of \mathbf{C} is a map of cooperads $\eta : \mathbf{I} \rightarrow \mathbf{C}$.

The decomposition map Δ induces a decomposition map $\tilde{\Delta} : C \rightarrow C \circ C$ realized by $\Delta - \eta \mathbf{I} \circ \text{id} + \eta \epsilon \circ \eta \mathbf{I}$.

The notation $\tilde{\Delta}^n$ is used for the map $C \rightarrow C^{\circ n+1}$ given by composition of the maps

$$(\tilde{\Delta} \circ \underbrace{\text{id} \circ \dots \circ \text{id}}_{n-1}) : C^{\circ n} \rightarrow C^{\circ n+1}.$$

A coaugmented cooperad \mathbf{C} is *conilpotent* if for every element $c \in \mathbf{C}$, there is a natural number N such that $\Delta^N c = (\Delta^{N-1} c) \circ \epsilon \mathbf{I}$.

Example 2.4. • The motivating example of an operad is the *endomorphism operad* of a vector space B , denoted $\text{End } B$. The module $(\text{End } B)_n$ is

$\text{Hom}(B^{\otimes n}, B)$ and the image of the unit is the identity map of B . Composition is given by composition of maps among tensor powers of B .

- The category of modules is a full subcategory of the category of cooperads (or operads) where for a module M , the cooperad \mathbf{M} has $M_0 = M$, $M_1 = I_1$, and only trivial compositions.
- The coassociative cooperad has M_n equal to the ground field for all n with every decomposition map induced by the canonical isomorphism between the ground field and its tensor powers.

Definition 2.5. Operads have a forgetful functor to collections whose left adjoint is called the *free* operad on a collection.

Conilpotent coaugmented cooperads have a forgetful functor to collections whose right adjoint is called the *cofree* cooperad on a collection (suppressing conilpotence and coaugmentation).

Both the cofree and free functor on M can be realized at the collection level as the collection of rooted planar trees with vertices decorated by elements of M , denoted $\mathcal{T}(M)$. This implies the following.

- (1) Fix an operad \mathbf{P} (with underlying collection P) and an element of $\mathcal{T}(M)$. That is, take a planar rooted tree T and an element of $\mathbf{P}(n)$ for every vertex of T of valence $n + 1$ (collectively called *a decoration of T by \mathbf{P}*). Then there is a canonical element of \mathbf{P} called *the composition of the decoration* induced by the counit of the forgetful free adjunction $\mathcal{F}(P) \xrightarrow{\epsilon} \mathbf{P}$. Since this is a monad, this operation is associative, in the sense that this composition can be done subtree by subtree and the output is insensitive to the choice of subtrees or order of composition.
- (2) Dually, given a cooperad \mathbf{C} with underlying collection C , the unit of the cofree forgetful adjunction $\mathbf{C} \xrightarrow{\eta} \mathcal{F}^c(C)$ yields the following. For every planar rooted tree T with n leaves and vertices $\{v_i\}$ where v_i has valence $n_i + 1$, and every element $c \in \mathbf{C}(n)$, there is a canonical set of elements $c_i \in \mathbf{C}(n_i)$. This procedure is called *the decomposition of c into a decoration of T by \mathbf{C}* . It can be realized as follows. Let $\tilde{\Delta}^N c$ stabilize as in the definition of conilpotence. Each summand corresponds to a tree with levels and decorations. Forget the levels and any decorations by \mathbf{I} and project onto the summand corresponding to the tree T . See [LV12, 5.8.7] for more details.
- (3) There is a canonical linear isomorphism ψ between the free operad on a collection and the conilpotent cofree cooperad on the same collection.

Lemma 2.6. *Let M be a collection. An endomorphism $\mathcal{F}^c(M) \rightarrow \mathcal{F}^c(M)$ is an isomorphism if and only if its restriction $M \subset \mathcal{F}^c(M) \rightarrow M$ is an isomorphism of collections.*

This is in precise parallel to the situation with power series, where a power series is invertible if and only if its constant term is invertible.

Proof. For F an endomorphism, let F_r denote its restriction. Note that $(F \circ G)_r = F_r \circ G_r$, which implies that if F is an isomorphism, so is F_r . On the other hand, if F_r is an isomorphism, then induction on the number of vertices in a tree in $\mathcal{F}^c(M)$ allows one to build an inverse F^{-1} . \square

Definition 2.7. Let \mathbf{P} be an operad with underlying collection P . The *canonical twist* $\Phi_{\mathbf{P}} : \mathcal{F}^c(P) \rightarrow \mathcal{F}^c(P)$ is the cooperad map induced by the composition $\phi_{\mathbf{P}} = \epsilon \circ \psi$:

$$\mathcal{F}^c(P) \xrightarrow{\psi} \mathcal{F}(P) \xrightarrow{\epsilon} \mathbf{P} \rightarrow P.$$

Lemma 2.8. *Let \mathbf{P} be an operad. Then the canonical twist is an isomorphism.*

Proof. Restricted to P , the canonical twist is the identity. Then Lemma 2.6 implies the result. \square

3. PARTITIONS AND TREES

Definition 3.1. Let $[n]$ be an ordered set and let $\pi = (p_1, \dots, p_k)$ be a partition of it, so that $[n]$ is the disjoint union of the blocks p_i . Blocks in our partitions are always ordered so that $\min p_i < \min p_j$ whenever $i < j$. A partition π is *crossing* if there exist w and y in p_i and x and z in p_j (with $i \neq j$) such that $w < x < y < z$. A partition π is *non-crossing* if it is not crossing. The notation $NC(n)$ (respectively $NC_k(n)$) refers to the set of noncrossing partitions of $[n]$ (with k blocks). The unique partition with a single block is called the trivial partition.

Noncrossing partitions are important in combinatorics and there are many bi-indexed sets of combinatorial objects in canonical bijection with them. For our purposes, the following such bijection will be useful.

Lemma 3.2. *The set $NC_k(n)$ is in bijection with the set of planar rooted trees with n leaves and $k+1$ vertices (including the root) satisfying the conditions that*

- (1) *Every non-root vertex has at least one leaf attached to it, and*
- (2) *the root has no leaves attached to it.*

The bijection from trees to partitions is given explicitly by numbering the leaves clockwise starting from the root and then letting two numbers share a block if the corresponding leaves are incident on the same vertex. Thus blocks are in bijection with non-root vertices.

See Figure 1. Henceforth partitions will be freely identified with the corresponding trees. It will be useful later to modify this construction.

Construction 3.3. Let π be a noncrossing partition of $[n]$. There is a function $h : \{0, \dots, n\} \rightarrow \{*, B_1, \dots, B_k\}$ defined by letting $h(i)$ be the maximal block (should one exist), which contains elements x and y of $[n]$ such that $x \leq i$ and $y > i$. Should no such block exist, then $h(i) = *$.

Now let i_0, \dots, i_n be non-negative numbers. The tree π_{i_0, \dots, i_n} is obtained from π by the following procedure:

- (1) For each j , attach i_j new leaves at the vertex $h(j)$ of π (if $h(j) = *$, attach to the root vertex) in the unique possible way so that the new leaves are after the j th original leaf and before the $j+1$ st original leaf of π , using the clockwise order around leaves.
- (2) Consider a non-root vertex v . Its incoming half-edges are of the form $e_{0,1}, \dots, e_{0,k_0}, \ell_1, e_{1,1}, \dots, e_{1,k_1}, \ell_2, \dots, \ell_r, e_{r,1}, \dots, e_{r,k_r}$, where ℓ_i are original leaves and $e_{i,j}$ are new leaves or parts of edges.

Let S_- be the set of indices j in $\{0, \dots, r\}$ such that $k_j \geq 1$ and let S_+ be the set of indices j such that $k_j > 1$. Now replace v with a tree which has

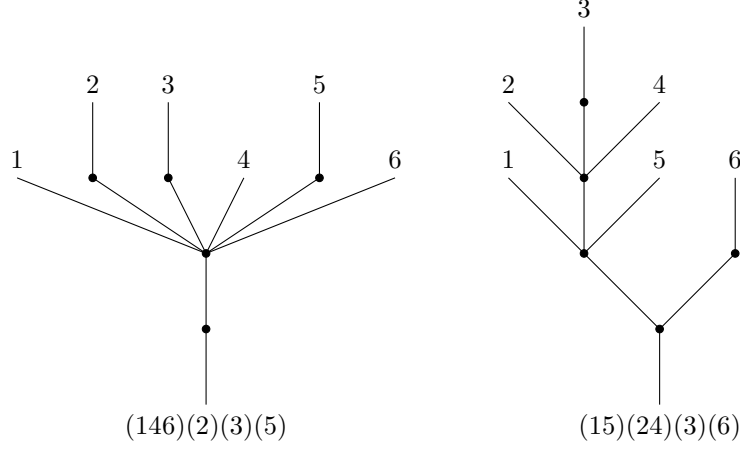


FIGURE 1. Two non-crossing partitions and the corresponding planar rooted trees

- (a) one “bottom” vertex with incoming half-edges in ordered bijection with S_- and
- (b) “top” vertices in ordered bijection with S_+ where the vertex v_j has k_j incoming half-edges.

Join the root of a top vertex with the corresponding incoming half-edge of the bottom vertex; the other incoming half-edges of the bottom vertex and all incoming half-edges of the top vertex are identified with the incoming half-edges of v .

- (3) Delete all of the original leaves of π ; delete the root if it is bivalent at this point in the construction.

The tree obtained after the intermediate step 2b will also be important and will be called $\bar{\pi}_{i_0, \dots, i_n}$.

See Figure 2.

Remark. Note that the first and last incoming half-edge at each “bottom” vertex of $\bar{\pi}_{i_0, \dots, i_n}$ are always original leaves of π .

4. FREE PROBABILITY AND OPERATOR-VALUED FREE CUMULANTS

Definition 4.1. Let B be an algebra. A B -valued probability space consists of a pair (A, E) where A is a B -algebra and E is a B -linear map, called *expectation* $A \rightarrow B$ such that the composition $B \rightarrow A \rightarrow B$ is the identity. By abuse of notation, E will usually be omitted.

Classically B is the ground field but for a general theory it is necessary to allow more general algebras, in particular, non-commutative algebras. To be precise, a B -algebra is a B -bimodule A equipped with a product $A \otimes_B A \rightarrow A$ and a B -linear map $\eta : B \rightarrow A$ which respects the product structure.

Let A be a B -valued probability space and let $f : A^{\otimes_B n} \rightarrow B$ be a B -multilinear map. For an $(n+1)$ -tuple (i_0, \dots, i_n) of non-negative integers with sum N , define a map

$$f_{i_0, \dots, i_n} : \text{Hom}(A^{\otimes n}, \text{Hom}(B^{\otimes N}, B))$$

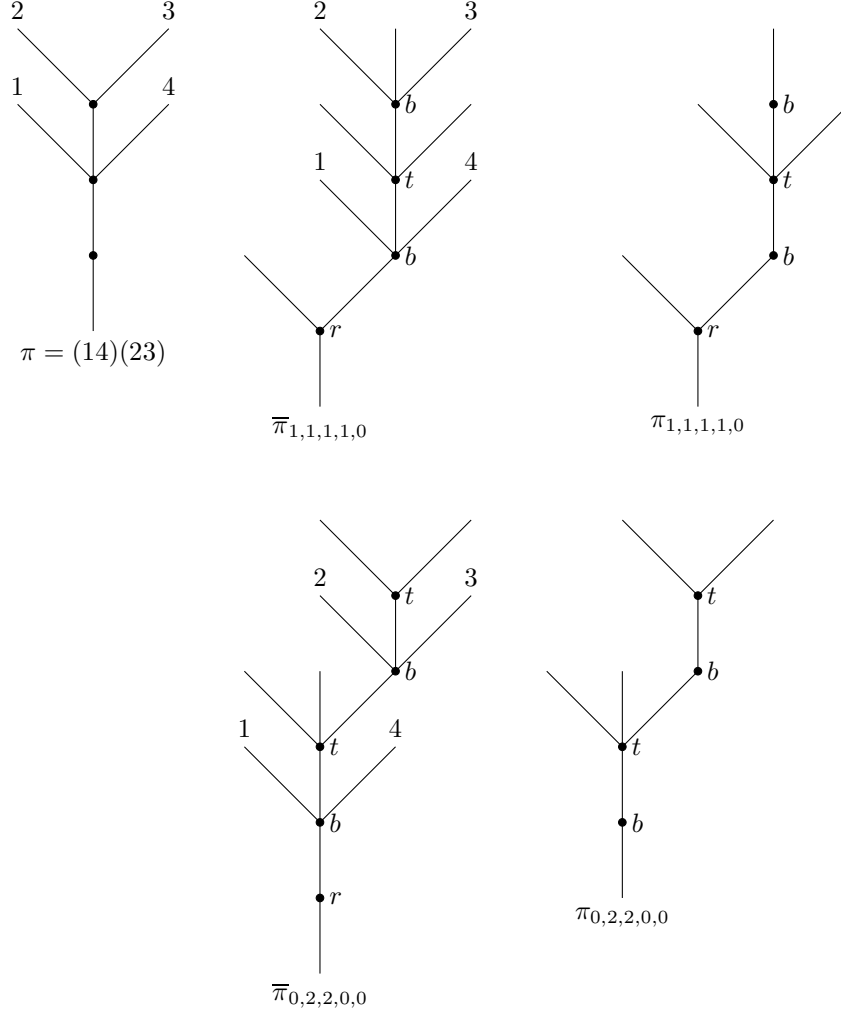


FIGURE 2. Examples of Construction 3.3. Top, bottom, and root vertices are labelled t , b , and r respectively.

whose evaluation on $a_1 \otimes a_n$ is given by the composition

$$B^{\otimes N} \rightarrow B^{\otimes i_0} \otimes A \otimes B^{\otimes i_1} \otimes A \otimes \cdots \otimes A \otimes B^{\otimes i_n} \rightarrow A^{\otimes_{B^n}} \xrightarrow{f} B.$$

Where the first map inserts a_j in the j th A place and the second map is given by repeated use of the B -bimodule structure on A .

The map $f_{i_0, \dots, i_n}(a_1, \dots, a_n)$ can be realized as the composition in $\text{End } B$ along a decoration of the tree $\pi_{\min\{i_0, 1\}, \dots, \min\{i_n, 1\}}$, where π is the trivial partition of $[n]$. Decorate “top” vertices and the root, should it exist, with the product in B and decorate the single vertex corresponding to the single block of π with $f_{\min\{i_0, 1\}, \dots, \min\{i_n, 1\}}(a_1, \dots, a_n)$.

The following definition is Definition 2.1.1 of [Spe98], restricted to B -algebras. It has been reworded to use operadic language.

Definition 4.2. Let A be a B -algebra. For $n \geq 1$, let $f^{(n)} : A^{\otimes_B n} \rightarrow B$ be a B -linear map. Then the *multiplicative function*

$$\hat{f} : \bigcup_n NC(n) \times A^{\otimes_B n} \rightarrow B$$

is defined on $(\pi, a_1 \otimes \cdots \otimes a_n)$ as the composition in $\text{End } B$ along a decoration of the tree $\pi_{0,\dots,0}$. “Top” vertices and the root vertex of π , if it survives in $\pi_{0,\dots,0}$, are decorated with the product in B . Let v be a “bottom” vertex of $\pi_{0,\dots,0}$ with ordered incoming half-edge set

$$\ell_0, e_{1,1}, \dots, e_{1,k_1}, \ell_1, \dots, \ell_{k-1}, e_{k,i_k}, \ell_k$$

where ℓ_i is an original leaf of π which is numbered $n(\ell_i)$ in π and i_j are non-negative numbers. Then the corresponding “bottom” vertex of $\pi_{0,\dots,0}$ is decorated with

$$f_{0,i_1,\dots,i_k,0}^{(k)}(a_{n(\ell_0)} \otimes \cdots \otimes a_{n(\ell_k)}).$$

Then $\hat{f}(\pi, a_1 \otimes \cdots \otimes a_n)$ is the composition of this decoration of $\pi_{0,\dots,0}$ in the operad $\text{End } B$, viewed as an element of $(\text{End } B)(0) \cong B$.

See Figure 3. The following definition combines Example 1.2.2, Definition 2.1.6, and Proposition 3.2.3 of [Spe98].

Definition 4.3. Let A be a B -valued probability space. The free cumulant $\kappa_n : A^{\otimes_B n} \rightarrow B$ is defined recursively in terms of the expectation as follows:

$$E(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \hat{\kappa}(\pi, a_1 \otimes \cdots \otimes a_n)$$

It will be useful in the next section to record a version of this defining relationship viewed in $\text{End } B$. The following is a direct application of the definitions.

Lemma 4.4. *The moment and cumulant satisfy the following relations for non-negative i_0, \dots, i_n and a_1, \dots, a_n in A :*

$$E_{i_0,\dots,i_n}(a_1, \dots, a_n) = \left(\sum_{\pi \in NC(n)} \hat{\kappa}(\pi, \bullet) \right)_{i_0,\dots,i_n} (a_1, \dots, a_n).$$

which in turn is the composition in $\text{End } B$ along the decorated tree π_{i_0,\dots,i_n} with decoration as in Definition 4.2.

5. MAIN RESULT

Definition 5.1. The cooperad coAs_A is the categorical product of the coassociative cooperad and the cooperad which is the algebra A concentrated in arity 0.

Let $V = \langle * \rangle$ be a one-dimensional free module. There is an explicit presentation of the sum of the modules of the underlying collection of coAs_A as $\bigoplus_{n=1}^{\infty} (A \oplus V)^{\otimes n}$. Here the arity n module consists of those elements that are degree n in the generator $*$ of V .

In this presentation, the cocomposition map is given as follows. Let w be a word in $*$ and A ; let $F(W)$ be the set of all ways of writing w as the concatenation $b_0 a_1 b_1 \cdots a_n b_n$ where

- The words b_i are (possibly empty) words in A
- The words a_i are nonempty words in $*$ and A .

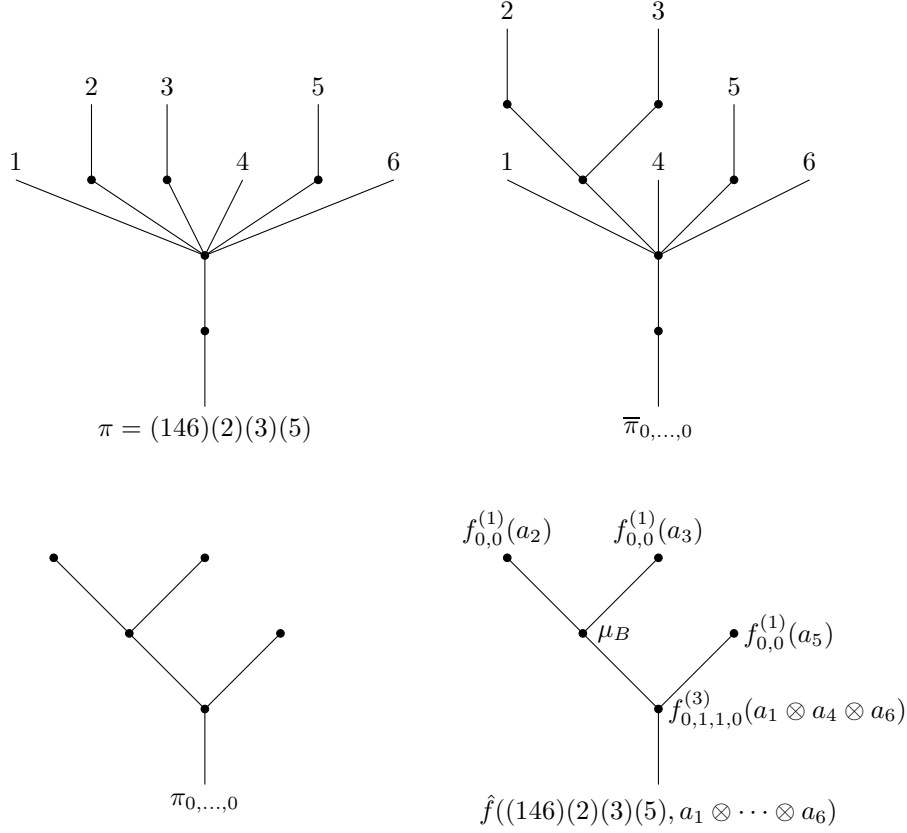


FIGURE 3. This figure demonstrates the evaluation of the multiplicative function f on the element

$$(146)(2)(3)(5), a_1 \otimes \dots \otimes a_6.$$

The eventual output is

$$f^{(3)}(a_1 f^{(1)}(a_2) f^{(1)}(a_3) \otimes a_4 f^{(1)}(a_5) \otimes a_6).$$

Then

$$\Delta w = \sum_{F(W)} (b_0 * b_1 * \dots * b_n) \circ (a_1 \otimes \dots \otimes a_n).$$

The projection map to coAs is given by projecting to $\bigoplus V^{\otimes n}$ and identifying $*^{\otimes n}$ with 1 in the ground ring. The projection map to A is given by projecting to $A \oplus V^{\otimes 1}$, identifying A with itself and $*$ with the image of I .

Definition 5.2. Let A be a B -valued probability space. The *moment morphism* $M : \text{coAs}_A \rightarrow \mathcal{F}^c(\text{End } B)$ is the map of cooperads determined by its linear restriction $m : \text{coAs}_A \rightarrow \text{End } B$ which is defined on the word

$$w = \underbrace{* \dots *}_{i_0} a_1 \underbrace{* \dots *}_{i_1} a_2 \dots a_m \underbrace{* \dots *}_{i_m}$$

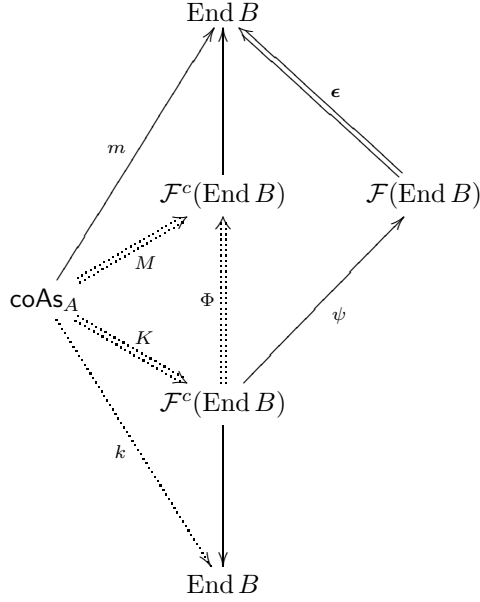


FIGURE 4. This figure shows the relationship between moments and cumulants. In the diagram, single arrows are linear maps and double arrows are cooperad and/or operad maps.

with $\sum i_j = n$ as

$$m(w)(b_1 \otimes \cdots \otimes b_n) = (E \circ \mu_A)_{i_0, \dots, i_m}(a_1 \otimes \cdots \otimes a_m)$$

or more explicitly

$$m(w)(b_{0,1} \otimes \cdots \otimes b_{m,i_m}) = E(b_{0,1} \cdots b_{0,i_0} a_1 b_{1,1} \cdots b_{1,i_1} a_2 \cdots a_m b_{m,1} \cdots b_{m,i_m}).$$

The *free cumulant morphism* $K : \text{coAs}_A \rightarrow \mathcal{F}^c(\text{End } B)$ is the map of cooperads $\Phi^{-1} \circ M$.

See Figure 5.

Theorem 5.3 (Main Result). *Let A be a B -valued probability space.*

The restriction k of the free cumulant morphism K consists of the following:

- (1) *The map $k(\underbrace{* \dots *}_n)$ is $(-1)^n$ times multiplication $B^{\otimes n} \rightarrow B$ for $n > 1$.*
- (2) *Let $\alpha_1, \dots, \alpha_{m-1}$ be in $\{0, 1\}$. Then*

$$k(a_1 \underbrace{* \dots *}_{\alpha_1} a_2 \cdots a_{m-2} \underbrace{* \dots *}_{\alpha_{m-1}} a_m)$$

(this description is slightly misleading because $\alpha_j \in \{0, 1\}$) is

$$(\kappa_m)_{0, \alpha_1, \dots, \alpha_{m-1}, 0}(a_1 \otimes \cdots \otimes a_m)$$

(in particular $k(a_1 \dots a_m) = \kappa(a_1 \otimes \cdots \otimes a_m)$).

- (3) *Applied to any word which contains an element of A and also two consecutive $*$ or a word beginning or ending with $*$ which contains an element of A , the map k vanishes.*

Proof. For the duration of the proof, let w be a word in $\overline{\text{coAs}}_A(n)$ which contains precisely the m letters (in order) a_1, \dots, a_m from A and n $*$ symbols, say

$$w = \underbrace{* \dots *}_{i_0} a_1 \underbrace{* \dots *}_{i_1} a_2 \dots a_m \underbrace{* \dots *}_{i_m}$$

with $i_j \geq 0$ and $\sum i_j = n$.

We give this word weight $2m - 1 + \sum i_j$. Weight is nonnegative, positive on the cokernel of the coaugmentation, and preserved by the decomposition map of coAs_A . The proof will proceed by induction on weight in each case.

Using the equivalent characterization $M = \Phi \circ K$ gives a recursive definition of k on the weight L component in terms of the value of m on the weight L component and the value of k on components of strictly smaller weight. That is, since the codomain $\mathcal{F}^c(\text{End } B)$ is cofree, it suffices to project to $\text{End } B$ for the definition, which yields $m = \phi \circ K$. Then via [LV12, Prop. 5.8.6] (or more properly speaking, its nonsymmetric version), the map K can be written at the level of collections as the composition

$$\text{coAs}_A \xrightarrow{\eta} \mathcal{T}(\text{coAs}_A) \xrightarrow{\mathcal{T}(k)} \mathcal{T}(\text{End } B)$$

Since ϕ is just $\epsilon \circ \psi$, and ψ is the identity at the level of collections, we can the relationship between m and k as the commutativity of the following diagram:

$$(2) \quad \begin{array}{ccc} \text{coAs}_A & \xrightarrow{m} & \text{End } B \\ \eta \downarrow & & \uparrow \epsilon \\ \mathcal{T}(\text{coAs}_A) & \xrightarrow{\mathcal{T}(k)} & \mathcal{T}(\text{End } B) \end{array}$$

where η and ϵ are the canonical decomposition and composition maps.

Now $\eta(w)$ consists of a sum of trees decorated with elements of $\overline{\text{coAs}}_A(j)$ for $j \leq n$. There is one summand corresponding to a tree with a single vertex decorated by w itself. This summand will contribute $k(w)$ to the eventual equation, and it is the purpose of the inductive step to determine its value. We will call this summand the *trivial* summand and call the tree T_{triv} .

We call decorated trees with a vertex decoration which contains an element of A and either the string $**$ or the symbol $*$ at the beginning or end of the decoration *degenerate*. In the first and second case of the statement of the theorem, by the inductive premise, only summands corresponding to nondegenerate decorated trees can contribute to $\eta(w)$. In the third case, T_{triv} is the only degenerate decorated tree that may contribute.

Any nondegenerate decorated tree T induces a partition π_T of $[m]$ by saying p and q are in the same block if a_p and a_q are in the same vertex decoration. This partition is necessarily non-crossing because the original tree was planar.

For π a partition, let \mathcal{T}_π be the set of nondegenerate decorated trees T such that $\pi_T = \pi$. Then the sum calculating $\eta(w)$ polarizes into subsums:

$$\sum_T \eta(w)_T = \sum_{\pi \in NC(m)} \sum_{T \in \mathcal{T}_\pi} \eta(w)_T$$

in the first two cases in the statement of the theorem (in fact, in the first case m is always 0), and

$$\sum_T \eta(w)_T = T_{\text{triv}} + \sum_{\pi \in NC(m)} \sum_{T \in \mathcal{T}_\pi} \eta(w)_T$$

in the third case of the statement of the theorem.

Then given a partition π arising from a nondegenerate decorated tree in the sum, the set of vertex decorations of $T \in \mathcal{T}_\pi$ which contain a letter of A is independent of T . This set of vertex decorations can be recovered from π as follows. Let a_{i_0}, \dots, a_{i_j} be a block of π . Then necessarily the vertex decoration is of the form $a_{i_0} *^{\alpha_1} \dots *^{\alpha_j} a_{i_j}$ where each α_p is either 0 or 1. If a_{i_ℓ} and $a_{i_{\ell+1}}$ are adjacent in w , then necessarily $\alpha_\ell = 0$. On the other hand, if a_{i_ℓ} and $a_{i_{\ell+1}}$ are separated in w , then necessarily $\alpha_\ell = 1$.

Fix a non-crossing partition π . Then there is a unique decorated tree in T_π with a minimal number of vertices, obtained as a decoration of π_{i_0, \dots, i_n} . The “bottom” vertices of this tree are decorated by the unique decoration described in the previous paragraph and all other vertices are decorated by $* \dots *$.

Then it is straightforward to verify that the set of nongenerate decorated trees T_π consists of all trees obtained from this decoration of π_{i_0, \dots, i_n} by replacing a vertex decorated by $* \dots *$ by a tree of the same overall arity, all of whose vertices are at least trivalent, and all of whose vertices are decorated by $* \dots *$.

At this point, it may be better to split into cases corresponding to the cases in the statement of the theorem.

- (1) On the word $w_n = (\underbrace{* \dots *}_n)$ with $n > 1$, the canonical decomposition $\eta(w_n)$

is then the sum over all planar rooted trees with n leaves; for each such tree the labels are all $* \dots *$. On all nontrivial trees, $\epsilon \circ \mathcal{T}(k)$ is, up to sign, just multiplication $\mu_n : B^{\otimes n} \rightarrow B$. Then by induction we have the formula

$$\mu_n = m(w_n) = k(w_n) + \sum \varepsilon_T \mu_n$$

In fact the indexing set of trees T for the sum is in canonical bijection with the non-top cells of the n -dimensional associahedron, and the sign ε_T is just the dimension of the corresponding cell, so this is essentially a sum which calculates the Euler characteristic of the associahedron. The associahedron is contractible so we get

$$\mu_n = k(w_n) + \mu_n(1 - (-1)^n).$$

- (2) Each individual set of trees \mathcal{T}_π has its summands in bijection, as in the previous case, with faces of associahedra. To be precise, in this case there is a product of associahedra, one for each vertex of π_{i_0, \dots, i_n} decorated by $* \dots *$. As in the previous case, the signs of k applied to these decorations are such that after applying ϵ to the subtrees where each vertex is decorated by $* \dots *$, what is obtained is a redecoration of the tree π_{i_0, \dots, i_n} , now by $\text{End } B$, as follows.

- (a) Vertices that were previously decorated by $* \dots *$ are now decorated by the product in $\text{End } B$, with no sign.
- (b) Vertices that were previously decorated by

$$a_{i_0} \underbrace{* \dots *}_{\alpha_1} a_{i_1} \dots a_{i_{j-1}} \underbrace{* \dots *}_{\alpha_j} a_{i_j}$$

(this description is slightly misleading because $\alpha_j \in \{0, 1\}$) are now decorated by induction by

$$(\kappa_j)_{0, \alpha_0, \dots, \alpha_j, 0}(a_{i_0} \otimes \dots \otimes a_{i_j})$$

as in the statement of the theorem, except for the following special case.

(c) the single tree T_{triv} is decorated by $k(w)$.

Then by Lemma 4.4, the equation $m(w) = \epsilon \circ \mathcal{T}(k) \circ \eta(w)$ is the same as the moment-cumulant formula for $E_{i_0, i_1, \dots, i_m, 0}(a_1 \otimes \dots \otimes a_m)$, up to the difference

$$k(w) - (\kappa_m)_{i_0, i_1, \dots, i_m}(a_1 \otimes \dots \otimes a_m).$$

So these two expressions are equal, as desired.

- (3) This is similar to the second case. Again, each set of trees \mathcal{T}_π is in bijection with faces of products of associahedra and by the same trick one obtains a redecoration of π_{i_0, \dots, i_n} . In this case the tree T_{triv} is not part of any \mathcal{T}_π but instead is its own separate summand. Then in this case the equation $m(w) = \epsilon \circ \mathcal{T}(k) \circ \eta(w)$ is the same as the moment-cumulant formula for $E_{i_0, i_1, \dots, i_m, 0}(a_1 \otimes \dots \otimes a_m)$, up to the difference

$$k(w)$$

so $k(w)$ is zero, as desired. □

Concluding remarks.

Remark. Equation (2) in the preceding proof suggests a different interpretation of the main result. The maps k and m can be understood as maps of collections between the underlying collection of the cooperad \mathbf{coAs}_A and the underlying collection of the operad $\mathbf{End} B$. The space of maps of collections from a cooperad to an operad possesses a rich natural structure (see [LV12, 6.4, 10.2.3] for details and notation). Apparently the relational equation can be expressed in terms of the convolution by the following expression:

$$m = \sum_{n=1}^{\infty} k^{\odot n} \approx \frac{k}{1 - k}.$$

Remark. The entire paper could be modified to work with symmetric operads; in this case it would be reasonable to replace \mathbf{coAs}_A with a commutative version \mathbf{coCom}_A . This would only make sense in the case that both A and B are commutative. As one might reasonably expect, this analagous procedure seems to describe the *classical* cumulants as the canonical twist of the moments. As there are many direct combinatorial presentations [RS00] for classical cumulants and even a significantly more direct approach from the operadic point of view [DCPT15b], any details or verification have been omitted.

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CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE (IBS), POHANG 37673, REPUBLIC OF KOREA